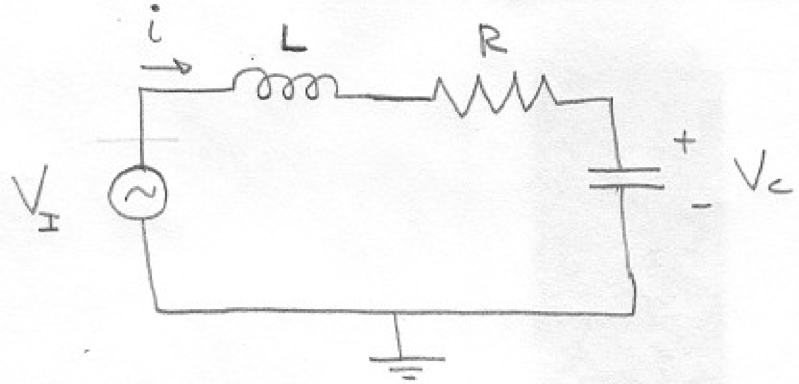
Second Order Systems

Let's consider a LRC circuit similar to that on page 1-3. This time an inductor is included that is governed by the equation $V_L = L \frac{di}{dt}$.



The voltage of the capacitor, resistor, and inductor must equal the voltage of the input.

$$V_I(t) = L\frac{di(t)}{dt} + R\,i(t) + \frac{1}{C}\int i(t)\,dt \qquad (1)$$

By definition current is the rate of change of charge

$\therefore \quad i(t) = \frac{dq(t)}{dt} \rightarrow$ making the substitution into (1)

$$V_I(t) = L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) \qquad (2)$$

Note that this is a second order differential eq. Also recall that for a capacitor $q(t) = C\,V_c(t)$ substituting into (2)

$$V_I(t) = LC\frac{d^2V_c(t)}{dt^2} + RC\frac{dV_c(t)}{dt} + V_c(t) \qquad (3)$$

Now let's take the Laplace transform of Eq. 3 (assuming no initial conditions)

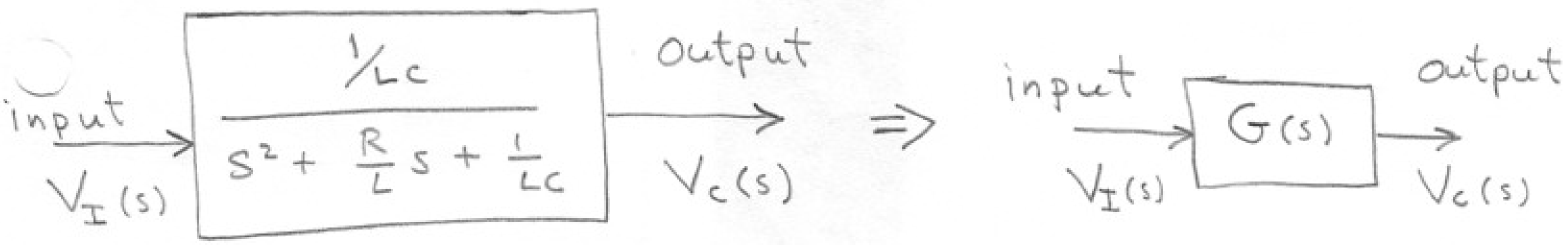
$$V_I(s) = s^2 LC\, V_c(s) + s RC\, V_c(s) + V_c(s) \qquad (4)$$

$$V_I(s) = V_c(s)\left\{ s^2 LC + sRC + 1 \right\}$$

We can rearrange terms to obtain the transfer function of the system.

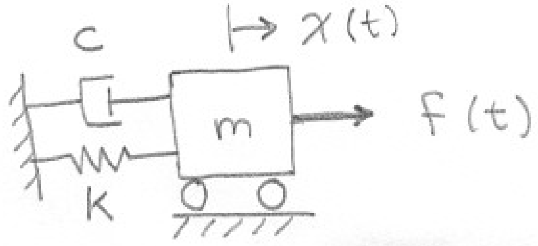
$$\frac{V_c}{V_I}(s) = \frac{1}{s^2 LC + sRC + 1} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

The system can be represented by a block diagram



Where $G(s)$ is the transfer function of the system. This is commonly referred to as the "frequency response function". The transfer function provides the relationship between the input voltage and the capacitor voltage. You can think of a transfer function as an amplifier gain that varies with frequency (ω). Note that the transfer function $G(s)$ is a complex quantity that varies with frequency.

Now let's consider a spring, mass, damper system with an external forcing function $f(t)$. The force represents the input to the system and the displacement, $x(t)$, represents the output of the system.



It can be shown that the equation of motion of the system is given by:

$$m \frac{d^2 x(t)}{dt^2} + c \frac{dx(t)}{dt} + K\, x(t) = f(t) \qquad (4)$$

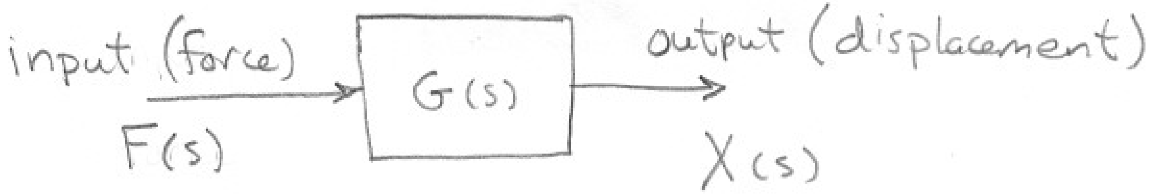
Taking the Laplace transform of Eq. 4 (assuming no initial conditions)

$$m s^2 X(s) + c s X(s) + K X(s) = F(s)$$

We can rearrange terms to obtain the transfer function of the system

$$\frac{\text{output}}{\text{input}} = G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + K} = \frac{1/m}{s^2 + \frac{c}{m}s + \frac{K}{m}}$$

The block diagram of the spring, mass, damper system is given by:



The general form of a second order system is given by:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where ω_n = natural frequency

ζ = damping ratio

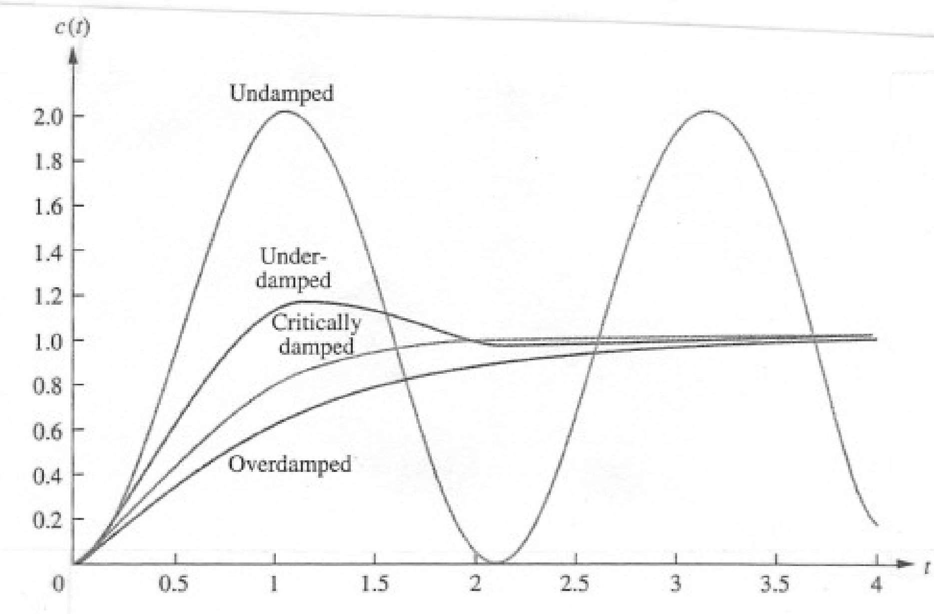
The unit step response of a second order system will depend on the natural frequency and the damping ratio. Three types of responses are possible

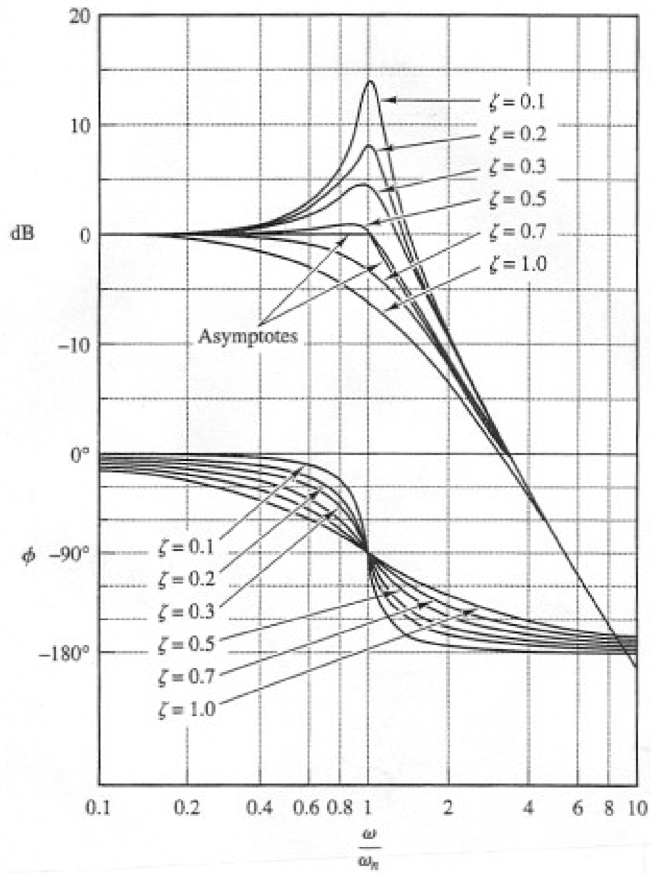
$\zeta < 1 \rightarrow$ underdamped, two complex poles

$\zeta = 1 \rightarrow$ critically damped, two repeated real poles

$\zeta > 1 \rightarrow$ overdamped, two distinct real poles

The typical step response for a typical second order system is shown below





A typical frequency response function (FRF) for a 2nd order system is shown above. The magnitude's shape will depend on its damping ratio. The peak occurs slightly lower in frequency than the natural frequency (ω_n). The location of the peak is the resonant frequency (ω_r). Notice that the phase of the output wrt. the input is initially zero, then at the natural frequency it's 90°; finally at high frequencies it's 180°. Lastly notice that well past the resonant frequency, the magnitude rolls off at 40 dB/decade. This is twice as fast as compared to first order systems (see p. 1-6)